Multipartite Entanglement for the Quantum Internet

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***Abstract*—Multipartite entanglement plays a crucial role in the Quantum Internet design, due to its potentiality of significantly increasing the network performance. In this paper, we identify the four key network functionalities for managing multipartite entanglement among remote nodes. And we discuss each functionality by considering – as case study – a specific multipartite state, which exhibits an attractive computing feature. Specifically, the designed state allows an arbitrary entangled node to calculate – in a distributed way – the sum of a set of values arbitrarily selected by the remaining entangled nodes.**

***Index Terms*—Quantum Internet; Entanglement; Multipartite; Quantum Communications; Quantum Networks.**

I. Introduction

The Quantum Internet is a global quantum network interconnecting multiple heterogeneous quantum networks [1]–[5], able to exchange quantum bits (qubits) and to generate end distribute entangled states.

A fundamental role in the Quantum Internet design is played by multipartite entanglement [6], [7], since it enables communication functionalities with no counterpart in the classical world. Yet, the generation and distribution of multipartite entanglement poses several challenges, ranging from quantum hardware technology [8] to quantum protocol stack design [6] through quantum vs. classical communication functionalities interplay [9].

From a communication engineering perspective, a key open issue is constituted by the quantum communication overhead induced by the management of multipartite entanglement among remote nodes. More into details, by abstracting from the particulars of the hardware devoted to entanglement generation, the communication overhead can be quantified as the amount of EPR pairs consumed by the quantum network functionalities to manage the entanglement relationship among multiple nodes.

From the above, it becomes evident that finding suitable multipartite states which minimize the number of required EPR pairs for the aforementioned management is a key strategy to mitigate the communication overhead.

In this paper, we identify four key network functionalities for managing multipartite entanglement in the Quantum Internet. We discuss the communication overhead associated with each function by considering, as case study, a specific multipartite entangled state, denoted as C^3 state, with the appealing property of exhibiting the lowest communication overhead for each key functionality.

Furthermore, the C^3 state exhibits an attractive computing property, which allows to calculate at a certain node of a multipartite entangled state the sum of a set of values arbitrarily selected by the remaining entangled nodes. Such a property can be exploited for designing advanced security functionalities in the Quantum Internet.

The paper is structured as follows. We first identify the four key network functionalities for multipartite entanglement in Section II. Then, we introduce the multipartite entangled state which exhibits the appealing performance in terms of four key functionalities as well as calculating functionality in Section III. Finally, we conclude the paper in Section IV.

II. Multipartite Entanglement: Key Network Functionalities

In order to distribute and manage multipartite entanglement in the Quantum Internet with communication overhead consideration, we identify four key network functionalities, as described in the following.

(1) Adding. Adding functionality is the primary factor that enables multipartite-entangled-based networks. It offers the possibility to enlarge set of entangled nodes in the quantum network by building entanglement relationship based on at least one pair of pre-shared EPR state between quantum nodes. For example, in Fig. 1, with the initial network consisting in two connected quantum nodes and one isolated node, adding functionality creates the quantum entangled link with each node so that the new multipartite entangled quantum network becomes interconnected.

(2) Deleting. Deleting functionality provides the possibility to dynamically eliminate entanglement relationship with target nodes in the Quantum Internet. For example, in Fig. 1, a three-parties completed entangled network can remove arbitrary node by deleting functionality. In general, this functionality is accomplished by quantum measurements and LOCC. With minimal or even no remote quantum operation, deleting functionality will contribute to mitigating communication overhead of Quantum Internet.

(3) Joining. The joining functionality extends the adding functionality for merging heterogeneous quantum sub-networks. For example, in Fig. 1, three independent quantum subnets can be interconnected by merging them into a 6-party full-connected entangled network, based on joining function-

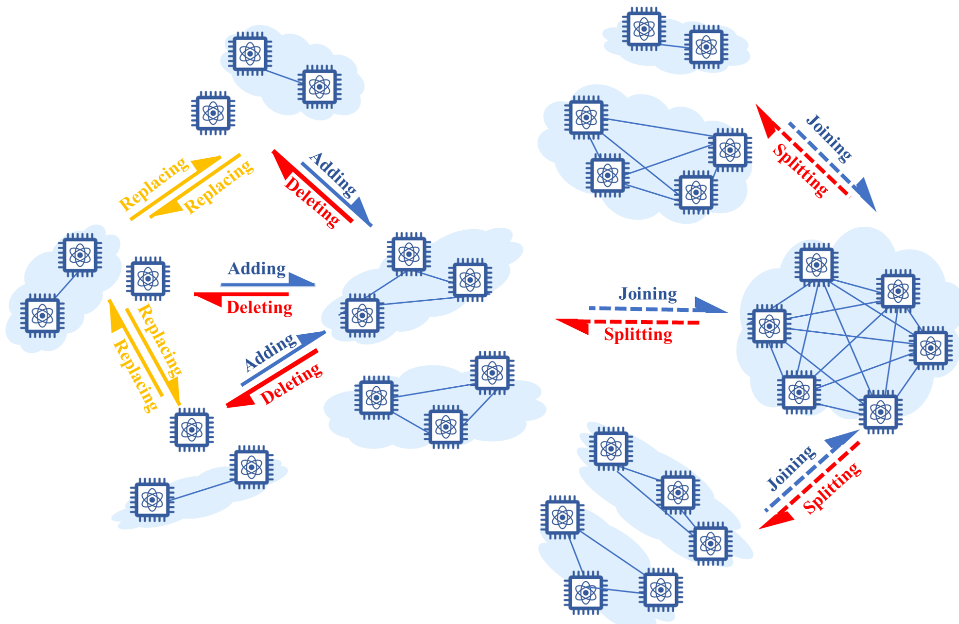

Fig. 1: Key network functionalities for distributing multipartite entanglement. Blue icons represent remote quantum nodes, whereas link between icons represent entanglement relationship. Cloud background denotes a set of quantum nodes sharing multipartite entanglement. *Adding* and *deleting* – denoted with blue and red solid-arrows respectively – enables the possibility to enlarge and to reduce the set of entangled nodes, i.e., to build and to eliminate entanglement relationship. *Joining* and *splitting* functions – denoted with blue and red dashed-arrows respectively – enables the possibility to merge and split entangled quantum networks. *Replacing* – denoted with yellow solid-arrow – enables the possibility to swap entanglement between quantum nodes.

ality. To this aim, generally a substantial amount of EPR pairs is consumed.

(4) Splitting. The splitting functionality extends the deleting functionality. Multipartite entanglement relationship is complex in the Quantum Internet so that it is hard to segment the networks adaptively. The splitting functionality supports adaptive network segmentation in the Quantum Internet. For example, a 6-party full-connected entangled network can be adaptively splitted in different subnets as depicted in Fig. 1.

We further note that by combing suitably the adding and deleting functionalities, a new functionality, named ***Replacing***, is obtained. In fact, node-specific replacing is equivalent to adding and then deleting. It enables the possibility to swap entanglement between quantum nodes. In Fig. 1, the different kinds of entanglement relationship among three quantum nodes can be swapped due to this functionality.

III. C^3 STATE

In this paper, we propose a multipartite entangled state, named C^3 state, which can be exploited for designing the four key functionalities described in Sec. II. To this aim, we first provide a recursive generation procedure for such a state, which is pivotal in distributed approaches. Then we show that such a state exhibits the peculiar property of consuming at most one EPR pair for the joining and splitting functionalities. Hence, it exhibits the lowest communication overhead, and it can be utilized as premium entanglement resources to be distributed in the Quantum Internet. Finally we discuss an attractive computing property of the C^3 state, which allows to calculate at a certain node of a multipartite entangled state the sum of a set of values arbitrarily spread among the remaining entangled nodes. Such a property can be exploited for designing advanced security functionalities in the Quantum Internet.

Definition 1. **(C^3 state)** The n-qubit C^3 state is defined as:

$$|C_n^3\rangle = \prod_{i=2,j=Random(i)}^{n} CNOT_{A_i,A_j} CZ_{A_i,A_j} |-\rangle^{\otimes n-1} |0\rangle_{A_1} \qquad (1)$$

where $(n-1)$ qubits $A_2, A_3, ..., A_n$ are initialized in state $|-\rangle$, and one single qubit denoted as A_1 without loss of generality, is initialized in state $|0\rangle$. Then C_n^3 state is obtained recursively by CZ and $CNOT$.

Remark. We use graph state notation [10] to describe C^3 state. More in detail, we give several corresponding graph illustrations in Fig. 2 to explain the generation of C^3 state. Specifically, we uniformly use n-vertex full connected graph in blue to represent C_n^3 state in following figure.

We introduce a slightly modification of C^3 state, denoted $\tilde{C}^3$, which is its orthogonal state, been LU equivalent to C^3.

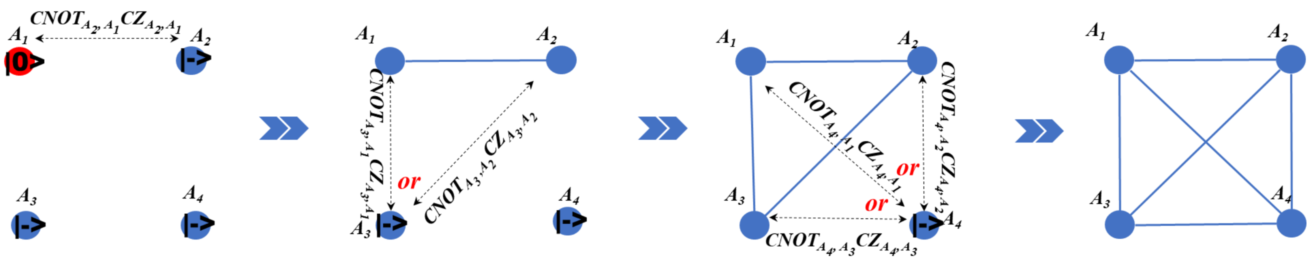

Fig. 2: Generating C_2^3, C_3^3, C_4^3 states. Qubits $A_2, A_3, ..., A_n$ are initialized in state $|-\rangle$, while qubit A_1 is initialized in state $|0\rangle$. (1) By performing $CNOT_{A_2,A_1}CZ_{A_2,A_1}$, qubits A_1, A_2 entangled into C_2^3 state. (2) Choosing an arbitrary qubit in C_2^3 with qubit A_3 to be performed on recursive CZ and $CNOT$ operations. (i.e. $CNOT_{A_3,A_1}CZ_{A_3,A_1}$ or $CNOT_{A_3,A_2}CZ_{A_3,A_2}$), qubit A_3 been entangled to C_3^3. (3) Similarly, choosing an arbitrary qubit in C_3^3 with qubit A_4 to be performed on recursive CZ and $CNOT$ operations. (i.e. $CNOT_{A_4,A_1}CZ_{A_4,A_1}$ or $CNOT_{A_4,A_2}CZ_{A_4,A_2}$ or $CNOT_{A_4,A_3}CZ_{A_4,A_3}$), then qubit A_4 been entangled to C_4^3. Here n-vertex full connected graph in blue represents C_n^3 state.

Definition 2. **($\tilde{C}^3$ state)** The n-qubits $\tilde{C}^3$ state is defined as:

$$\left|\tilde{C}_n^3\right\rangle = \prod_{i=2,j=Random(i)}^{n} CZ_{B_i,B_j}CNOT_{B_i,B_j}\left|+\right\rangle^{\otimes n-1}\left|1\right\rangle_{B_1} \tag{2}$$

Similarly to Definition 1, $(n-1)$ qubits $B_2, B_3, ..., B_n$ are initialized in state $|+\rangle$, and one single qubit denoted as B_1 without loss of generality, is initialized in state $|1\rangle$. Then $\tilde{C}^3$ state is obtained recursively by $CNOT$ and CZ.

Remark. As already mentioned, the two state are LU-equivalent, since $\left|\tilde{C}^3\right\rangle = X_iZ_i\left|C^3\right\rangle = X_i\prod_{j=1, j\neq i}^{n} Z_j\left|C_n^3\right\rangle$, $\left|C^3\right\rangle = Z_pX_p\left|\tilde{C}^3\right\rangle = X_p\prod_{p=1, p\neq q}^{n} Z_q\left|\tilde{C}_n^3\right\rangle$, where i, j is arbitrary particle belongs to $\left|C^3\right\rangle$, and p, q is arbitrary particle belongs to $\left|\tilde{C}^3\right\rangle$ state. Hence $\tilde{C}^3$ owns the same properties as C^3 state. To simplify description, we will introduce the properties of C^3 regardless of $\tilde{C}^3$ in the following parts.

Lemma 1. ***(Recursive generation)*** *A n-qubit $\left|C_n^3\right\rangle$ state can be generated from k-qubit $\left|C_k^3\right\rangle$ state with $\left|-\right\rangle^{\otimes(n-k)}$ by recursively performing $(n-k)$ times of CU operation.*

From Definition 1, it follows that a n-qubit state $\left|C_n^3\right\rangle$ can be obtained from a $(n-1)$-qubit state $\left|C_{n-1}^3\right\rangle$ by simply performing a CU operation between qubit A_n initialized in state $|-\rangle$ with A_i, an arbitrary qubit of state $\left|C_{n-1}^3\right\rangle$, with the former acting as control and the latter acting as target of the CU gate given by:

$$CU = CNOT \otimes CZ = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \tag{3}$$

More in detail, it results:

$$\left|C_n^3\right\rangle = CU_{A_n,A_i}\left|-\right\rangle_{A_n} \otimes \left|C_{n-1}^3\right\rangle_{A_1,...,A_i,...,A_{(n-1)}} \tag{4}$$

In same way, $\left|C_{n-1}^3\right\rangle$ state can be generated from $\left|C_{n-2}^3\right\rangle$ with $|-\rangle$ by performing a CU operation. Therefore n-qubit $\left|C_n^3\right\rangle$ state can be generated from k-qubit $\left|C_k^3\right\rangle$ state with $\left|-\right\rangle^{\otimes(n-k)}$ by recursively performing $(n-k)$ times of CU operation.

Remark. **(Adding local qubit)** Given a n-qubit state $\left|C_n^3\right\rangle$, one qubit $n+1$, located in same node with an arbitrary qubit in $\left|C_n^3\right\rangle$, can be entangled into a $(n+1)$-qubit state $\left|C_{n+1}^3\right\rangle$ by local operation CU.

In order to add one local qubit into Quantum Internet, we can use C^3 state to realize it with lowest communication overhead. Suppose a n-qubit C^3 state is distributed among different nodes, the new qubit is just located the same node with an arbitrary qubit in C^3 state. From Equation 4, after performing CU operation on the new qubit with same located qubit, it obtains $(n+1)$-qubit C^3 state.

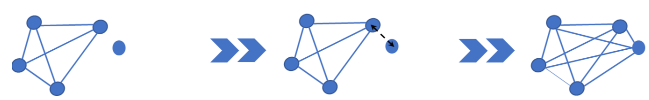

Fig. 3: Adding local qubit from C_4^3 to C_5^3 state. (1) New qubit (denoted in blue solid dot without link) is located at the same quantum node with an arbitrary qubit in C_4^3 state. (2) Performing CU operation, denoted as a black dash bidirectional arrow, on the new qubit with same located qubit. (3) Obtaining C_5^3 state denoted by 5-vertex full-connected graph.

More detail, we give an example of adding one local qubit from C_4^3 to C_5^3 state in Fig 3. Specifically, We uniformly use a black dash bidirectional arrow to represent CU operation in following figures.

Remark. **(Adding remote node)** Given a n-qubit state $\left|C_n^3\right\rangle$, one qubit $n+1$ located in remote node can be entangled to a $(n+1)$-qubit state $\left|C_{n+1}^3\right\rangle$ by consuming one EPR pair.

In order to add remote one node into Quantum Internet, we can use C^3 state to realize it with lowest communication overhead. Suppose a n-qubit C^3 state is distributed among different nodes, the new qubit $(n+1)$ is located at remote node been neighbour with a quantum node equipped at least one qubit in C^3 state.

We can pre-share one EPR pair $|\Phi^-\rangle_{i'n+1} = \frac{1}{\sqrt{2}}(|00\rangle - |11\rangle)_{i'n+1}$ between the remote node with its neighbouring node which equipped at least one qubit i in $\left|C_n^3\right\rangle$ state. The initial state is:

$$|\Phi_{initADDR}\rangle = \left|C_n^3\right\rangle_{1,\dots i,\dots,n} \left|\Phi^-\right\rangle_{i'n+1} \tag{5}$$

After performing $CU_{i',i}$ on qubit i' and an arbitrary qubit i of C^3 that located in neighbour node, the state becomes:

$$\begin{aligned} |\Phi_{ADDR}\rangle = &\frac{1}{\sqrt{2^{n+1}}}\left(\left|C_n^3\right\rangle|0\rangle_{n+1} - \left|\tilde{C}_n^3\right\rangle|1\rangle_{n+1}\right)|+\rangle_{i'} \\ &+\frac{1}{\sqrt{2^{n+1}}}\left(\left|C_n^3\right\rangle|0\rangle_{n+1} + \left|\tilde{C}_n^3\right\rangle|1\rangle_{n+1}\right)|-\rangle_{i'} \end{aligned} \tag{6}$$

Finally, performing quantum measurement on qubit i' under X-basis, the final state can be obtained as $\left|C_n^3\right\rangle|0\rangle_{n+1} - \left|\tilde{C}_n^3\right\rangle|1\rangle_{n+1}$, which is $\left|C_{n+1}^3\right\rangle$ proved in Lemma 2, by I or Z operation.

Here we give an example of adding remoter node from C_4^3 to C_5^3 state in Fig 4. Specifically, we uniformly use a pair of linked solid red dots to represent EPR state $|\Phi^-\rangle$ and use dash rectangle surrounding solid dot(s) to represents quantum measurement on that qubit(s) in the following figures.

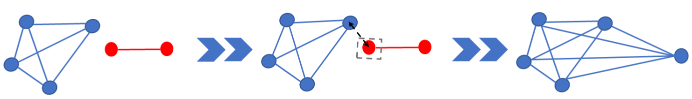

Fig. 4: Adding remote node from C_4^3 to C_5^3 state. (1) A pair of EPR distributed among new node (denoted in the right red dot located) and its neighbour quantum node (denoted in the left red dot with an arbitrary qubit in C_4^3 state located). (2) Performing CU operation on the qubit (denoted by left solid red dot) in EPR state with one qubit (denoted by solid blue dot in upper right corner) in C_4^3, which both at same node. Then Performing X-basis quantum measurement on the former qubit (denoted by left solid red dot). (3) Obtaining $\left|C_5^3\right\rangle$ by I or Z operation.

Remark. (Joining) Given a n-qubit state $\left|C_n^3\right\rangle$ and a m-qubit state $\left|C_m^3\right\rangle$, a $(n+m)$-qubit state $\left|C_{n+m}^3\right\rangle$ can be deterministically obtained by consuming one EPR pairs.

In order to join quantum network, we firstly find a pair of neighbour nodes respectively located in two pre-joined networks. Then we can pre-share one EPR pair $|\Phi^-\rangle_{12}$ between these two neighbouring nodes, which respectively equipped at least one qubit i in $\left|C_n^3\right\rangle$ state and one qubit j in $\left|C_m^3\right\rangle$. The initial state is:

$$|\Phi_{initJoin}\rangle = \left|C_n^3\right\rangle\left|\Phi^-\right\rangle_{12}\left|C_m^3\right\rangle \tag{7}$$

Then we perform $CU_{1,i}$ operation on qubit 1 and a qubit i of C_n^3 in one quantum node as well as $CU_{2,j}$ operation on qubit 2 and a qubit j of C_m^3 in anther one node. Then the state becomes:

$$\begin{aligned} |\Phi_{Join}\rangle = &\frac{1}{\sqrt{2^{n+m+1}}}\left(\left|C_n^3\right\rangle\left|C_m^3\right\rangle - \left|\tilde{C}_n^3\right\rangle\left|\tilde{C}_m^3\right\rangle\right)|+\rangle_1|+\rangle_2 \\ &+\frac{1}{\sqrt{2^{n+m+1}}}\left(\left|C_n^3\right\rangle\left|C_m^3\right\rangle + \left|\tilde{C}_n^3\right\rangle\left|\tilde{C}_m^3\right\rangle\right)|+\rangle_1|-\rangle_2 \\ &+\frac{1}{\sqrt{2^{n+m+1}}}\left(\left|C_n^3\right\rangle\left|C_m^3\right\rangle + \left|\tilde{C}_n^3\right\rangle\left|\tilde{C}_m^3\right\rangle\right)|-\rangle_1|+\rangle_2 \\ &+\frac{1}{\sqrt{2^{n+m+1}}}\left(\left|C_n^3\right\rangle\left|C_m^3\right\rangle - \left|\tilde{C}_n^3\right\rangle\left|\tilde{C}_m^3\right\rangle\right)|-\rangle_1|-\rangle_2 \end{aligned} \tag{8}$$

When we measure qubits 1,2 with X-basis, reminds $(n+m)$-qubits will be joined as $\left|C_{n+m}^3\right\rangle$ state, from Lemma 3, as the measure results are $|+\rangle_1|+\rangle_2$ or $|-\rangle_1|-\rangle_2$. While the measure results are $|+\rangle_1|-\rangle_2$ or $|-\rangle_1|+\rangle_2$, from Remark III, reminds $(n+m)$-qubits can be joined as $\left|C_n^3\right\rangle\left|\tilde{C}_m^3\right\rangle + \left|\tilde{C}_n^3\right\rangle\left|C_m^3\right\rangle$ by local operation, that is $\left|\tilde{C}_{n+m}^3\right\rangle$ mentioned in Lemma 3. We also can transform it into $\left|C_{n+m}^3\right\rangle$ by local operation.

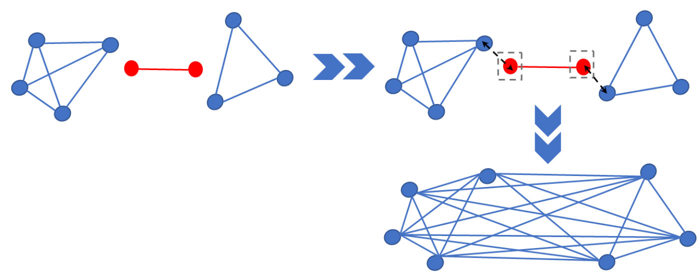

Fig. 5: Joining C_3^3 into C_4^3 quantum network. (1) A pair of EPR state is distributed among neighbouring nodes, respectively located in C_4^3 and C_3^3 quantum system. (2) Neighbouring nodes respectively perform local operation CU on the qubit in pre-shared EPR state with the local qubit in C^3 state. Then X-basis quantum measurement performed on two qubits initiated in EPR pair respectively. (3) Obtaining the completing C_7^3 quantum networks consisted of C_4^3 and C_3^3 quantum networks.

Here we give an example of joining C_3^3, C_4^3 into C_7^3 quantum network in Fig. 5.

Lemma 2. ***(Iteration)*** *A n-qubit state* $\left|C_n^3\right\rangle$ *or* $\left|\tilde{C}_n^3\right\rangle$ *can be entangled by (n-1)-qubit state* $\left|C_{n-1}^3\right\rangle$, $\left|\tilde{C}_{n-1}^3\right\rangle$ *with one qubit, which satisfies following iterative formula.*

$$\begin{cases} \left|C_n^3\right\rangle = \frac{1}{\sqrt{2^{n-1}}}\left(|0\rangle_{A_j}\left|C_{n-1}^3\right\rangle - |1\rangle_{A_j}\left|\tilde{C}_{n-1}^3\right\rangle\right) \\ \left|\tilde{C}_n^3\right\rangle = \frac{1}{\sqrt{2^{n-1}}}\left(|0\rangle_{B_j}\left|\tilde{C}_{n-1}^3\right\rangle + |1\rangle_{B_j}\left|C_{n-1}^3\right\rangle\right) \end{cases} \tag{9}$$

and

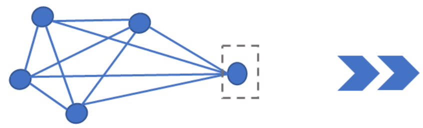

Fig. 6: Deleting one qubit from C_5^3 to C_4^3. Performing Z-basis quantum measurement (denoted by dash rectangular) on the target qubit (solid blue dot surrounded by dash rectangular), while remaining qubits maintains C_4^3 state.

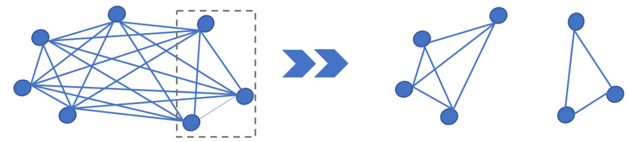

Fig. 7: Splitting C_7^3 into C_4^3 and C_3^3 state. Performing joint quantum measurement (denoted by dash rectangular) on target qubits (solid blue dots surrounded by dash rectangular), while remaining qubits maintains C_4^3 state.

$$\begin{cases} \left|C_n^3\right\rangle = \frac{1}{\sqrt{2^{n-1}}}\left(\left|C_{n-1}^3\right\rangle |0\rangle_{A_j} - \left|\tilde{C}_{n-1}^3\right\rangle |1\rangle_{A_j}\right) \\ \left|\tilde{C}_n^3\right\rangle = \frac{1}{\sqrt{2^{n-1}}}\left(\left|\tilde{C}_{n-1}^3\right\rangle |0\rangle_{B_j} + \left|C_{n-1}^3\right\rangle |1\rangle_{B_j}\right) \end{cases} \tag{10}$$

where $n \geq 2$ *and qubit* A_j, B_j *is an arbitrary qubit belonging to the original state* $\left|C_n^3\right\rangle$, $\left|\tilde{C}_n^3\right\rangle$ *respectively. For* $n = 1$, $\left|C_1^3\right\rangle = |0\rangle$, $\left|\tilde{C}_1^3\right\rangle = |1\rangle$.

Proof:
For $n = 2$,

$$\begin{cases} \left|C_2^3\right\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle \left|C_1^3\right\rangle - |1\rangle \left|\tilde{C}_1^3\right\rangle\right) \\ \left|\tilde{C}_2^3\right\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle \left|\tilde{C}_1^3\right\rangle + |1\rangle \left|C_1^3\right\rangle\right) \end{cases}$$

If $n = k$,

$$\begin{cases} \left|C_k^3\right\rangle = \frac{1}{\sqrt{2^{k-1}}}\left(|0\rangle \left|C_{k-1}^3\right\rangle - |1\rangle \left|\tilde{C}_{k-1}^3\right\rangle\right) \\ \left|\tilde{C}_k^3\right\rangle = \frac{1}{\sqrt{2^{k-1}}}\left(|0\rangle \left|\tilde{C}_{k-1}^3\right\rangle + |1\rangle \left|C_{k-1}^3\right\rangle\right) \end{cases}$$

is supposed to be corrected.
Then $n = k + 1$, from Lemma 1,

$$\begin{cases} \left|C_{k+1}^3\right\rangle = CU_{k+1,1} \left|-\right\rangle_{k+1} \left|C_k^3\right\rangle_{1,2,\ldots,k} \\ \qquad = \frac{1}{\sqrt{2^k}}\left(|0\rangle \left|C_k^3\right\rangle - |1\rangle \left|\tilde{C}_k^3\right\rangle\right) \\ \left|\tilde{C}_{k-1}^3\right\rangle = CU_{k+1,1}^T \left|+\right\rangle_{k+1} \left|\tilde{C}_k^3\right\rangle_{1,2,\ldots,k} \\ \qquad = \frac{1}{\sqrt{2^k}}\left(|0\rangle \left|\tilde{C}_k^3\right\rangle + |1\rangle \left|C_k^3\right\rangle\right) \end{cases}$$

is correct.

Therefore, $\forall n$ for Equation 9 can be valid. Similarly, Equation 10 can be proved too.

Remark. (Deleting) Deleting arbitrary m qubits in n-qubit C_n^3 state, remaining $(n-m)$-qubit maintains C_{n-m}^3 state.

In order to delete node or qubit in n-parties Quantum Internet entangled by C_n^3 state, we can easily perform Z-basis measurement on the quantum target. From Equation 9 10, any qubit in $\left|C_n^3\right\rangle$ state can be deleted with remain $(n-1)$ qubits in $\left|C_{n-1}^3\right\rangle$ state.

Here we give an example of deleting one qubit from C_5^3 to C_4^3 in Fig 6.

Lemma 3. ***(Complete bipartite state)*** *A* $(n+m)$*-qubit state* $\left|C_{n+m}^3\right\rangle$ *and* $\left|\tilde{C}_{n+m}^3\right\rangle$ *can be expanded as completed bipartite state:*

$$\begin{cases} \left|C_{n+m}^3\right\rangle = \frac{1}{\sqrt{2^{n+m-1}}}\left(\left|C_n^3\right\rangle \left|C_m^3\right\rangle - \left|\tilde{C}_n^3\right\rangle \left|\tilde{C}_m^3\right\rangle\right) \\ \left|\tilde{C}_{n+m}^3\right\rangle = \frac{1}{\sqrt{2^{n+m-1}}}\left(\left|C_n^3\right\rangle \left|\tilde{C}_m^3\right\rangle + \left|\tilde{C}_n^3\right\rangle \left|C_m^3\right\rangle\right) \end{cases} \tag{11}$$

Proof:
For $n = 1, m = 1$,

$$\begin{cases} \left|C_2^3\right\rangle = \frac{1}{\sqrt{2}}\left(\left|C_1^3\right\rangle \left|C_1^3\right\rangle - \left|\tilde{C}_1^3\right\rangle \left|\tilde{C}_1^3\right\rangle\right) \\ \left|\tilde{C}_2^3\right\rangle = \frac{1}{\sqrt{2}}\left(\left|C_1^3\right\rangle \left|\tilde{C}_1^3\right\rangle + \left|\tilde{C}_1^3\right\rangle \left|C_1^3\right\rangle\right) \end{cases}$$

If $n = k, m = l$,

$$\begin{cases} \left|C_{k+l}^3\right\rangle = \frac{1}{\sqrt{2^{k+l-1}}}\left(\left|C_k^3\right\rangle \left|C_l^3\right\rangle - \left|\tilde{C}_k^3\right\rangle \left|\tilde{C}_l^3\right\rangle\right) \\ \left|\tilde{C}_{k+l}^3\right\rangle = \frac{1}{\sqrt{2^{k+l-1}}}\left(\left|C_k^3\right\rangle \left|\tilde{C}_l^3\right\rangle + \left|\tilde{C}_k^3\right\rangle \left|C_l^3\right\rangle\right) \end{cases}$$

is supposed to be corrected.
Then $n = k + 1, m = l$, from Lemma 2,

$$\begin{cases} \left|C_{k+l+1}^3\right\rangle = \frac{1}{\sqrt{2^{k+l}}}\left(\left|C_1^3\right\rangle \left|C_{k+l}^3\right\rangle - \left|\tilde{C}_1^3\right\rangle \left|\tilde{C}_{k+l}^3\right\rangle\right) \\ \qquad = \frac{1}{\sqrt{2^{k+l}}}\left(\left|C_{k+1}^3\right\rangle \left|C_l^3\right\rangle - \left|\tilde{C}_{k+1}^3\right\rangle \left|\tilde{C}_l^3\right\rangle\right) \\ \left|\tilde{C}_{k+l+1}^3\right\rangle = \frac{1}{\sqrt{2^{k+l}}}\left(\left|C_1^3\right\rangle \left|\tilde{C}_{k+l}^3\right\rangle + \left|\tilde{C}_1^3\right\rangle \left|C_{k+l}^3\right\rangle\right) \\ \qquad = \frac{1}{\sqrt{2^{k+l}}}\left(\left|C_{k+1}^3\right\rangle \left|\tilde{C}_l^3\right\rangle + \left|\tilde{C}_{k+1}^3\right\rangle \left|C_l^3\right\rangle\right) \end{cases}$$

is correct.

Therefore, $\forall n, m$ for Equation 11 can be valid.

Remark. (Splitting) Splitting m-qubit C_m^3 state from $(n+m)$-qubit C_{n+m}^3 state, remaining n-qubit maintain C_n^3 state.

In order to split two arbitrary subnetwork from a complete Quantum Internet consisted of $(n+m)$-qubit C_{n+m}^3 state, with $n, m \geq 2$, we can perform joint measurement on arbitrary n qubits under quantum basis consists of $\left|C_n^3\right\rangle\left\langle C_n^3\right|$, $\left|\tilde{C}_n^3\right\rangle\left\langle \tilde{C}_n^3\right|$ and other n-dimensional orthometric observables. From Lemma 3, the n-qubits system is deterministically collapsed in $\left|C_n^3\right\rangle$ or $\left|\tilde{C}_n^3\right\rangle$ state, which can be transformed each other by X, Z operation based on Remark III. While remain m-qubit system also deterministically transforms to $\left|C_n^3\right\rangle$ or $\left|\tilde{C}_n^3\right\rangle$ as required. Here we give an example of splitting two quantum systems (C_4^3 and C_3^3) from C_7^3 in Fig. 7.

Remark. (Replacing) Given a n-qubit state $\left|C_n^3\right\rangle$, replace arbitrary qubit with new particle can be determistically performed by consuming only one EPR pair.

In order to replace an arbitrary qubit with new one in remote node into Quantum Internet, we can use C^3 state to realize it with lowest communication overhead. Suppose a n-qubit C_n^3 state is distributed among different nodes. The new remote pre-replaced node is neighbour with the quantum node equipped replaced qubit i in $\left|C_n^3\right\rangle_{1,2,\ldots i \ldots n}$ state. Here we give an example of replacing a qubit in C_4^3 with a new qubit located at remote node in Fig. 8.

Fig. 8: Replace one node in C_4^3 state with a new qubit located at remote node. (1) A pair of EPR state is distributed among replaced and target nodes. (2) Performing Bell measurement on replaced qubit with one of qubit in given EPR state at the replaced node. (3) Replacing new node into C^4 state.

Firstly, we preshare a pair of EPR state $|\Phi^-\rangle_{0i'}$ among these two nodes. The initial state is:

$$\begin{aligned} &|\Phi_{initRep}\rangle = \left|C_n^3\right\rangle_{1,\ldots i \ldots n} \left|\Phi^-\right\rangle_{0i'} \\ &= \frac{1}{\sqrt{2^{n+1}}} \left|\Phi^+\right\rangle_{i0} \left(\left|C_{n-1}^3\right\rangle |0\rangle_{i'} + \left|\tilde{C}_{n-1}^3\right\rangle |1\rangle_{i'}\right) \\ &+ \frac{1}{\sqrt{2^{n+1}}} \left|\Phi^-\right\rangle_{i0} \left(\left|C_{n-1}^3\right\rangle |0\rangle_{i'} - \left|\tilde{C}_{n-1}^3\right\rangle |1\rangle_{i'}\right) \\ &- \frac{1}{\sqrt{2^{n+1}}} \left|\Psi^+\right\rangle_{i0} \left(\left|\tilde{C}_{n-1}^3\right\rangle |0\rangle_{i'} + \left|C_{n-1}^3\right\rangle |1\rangle_{i'}\right) \\ &- \frac{1}{\sqrt{2^{n+1}}} \left|\Psi^-\right\rangle_{i0} \left(\left|\tilde{C}_{n-1}^3\right\rangle |0\rangle_{i'} - \left|C_{n-1}^3\right\rangle |1\rangle_{i'}\right) \end{aligned} \tag{12}$$

We can perform Bell measurement on i with the qubit 0 in given EPR pair $|\Phi^-\rangle$. From Equation 12 and Lemma 2, new $\left|C_n^3\right\rangle$ state replaced by qubit i' can be obtained.

Remark. (Calculating) The sum of a set of values, which randomly assigned on one or multi entangled qubits in C^3 state, can be calculated by an arbitrary entangled node in a distributed way.

In order to calculate the sum of a set of values, which randomly assigned on one or multi entangled nodes in the Quantum Internet, we can use C^3 to achieve it in a distributed way.

Suppose a n-qubit C_n^3 state is distributed among different nodes and randomly choose a aggregation node A_k equipped with qubit k from them. A set of values $\theta_1, \theta_2, \ldots, \theta_n$, which randomly assigned on each entangled nodes, can be encoded into C^3 state by rotation operation:

$$R_{\theta_i} = \begin{bmatrix} \cos\theta_i & -\sin\theta_i \\ -\sin\theta_i & -\cos\theta_i \end{bmatrix} \tag{13}$$

After performing R_{θ_i} on each qubit $1, 2, \ldots, n$, the whole quantum system becomes:

$$\begin{aligned} |\Phi_{Cal}\rangle = &\frac{1}{\sqrt{2^{n-1}}} \begin{bmatrix} \cos\left(\theta_1+\theta_2+\ldots+\theta_n\right)|0\rangle \\ -\sin\left(\theta_1+\theta_2+\ldots+\theta_n\right)|1\rangle \end{bmatrix}_k \left|C_{n-1}^3\right\rangle \\ &- \frac{1}{\sqrt{2^{n-1}}} \begin{bmatrix} \sin\left(\theta_1+\theta_2+\ldots+\theta_n\right)|0\rangle \\ +\cos\left(\theta_1+\theta_2+\ldots+\theta_n\right)|1\rangle \end{bmatrix}_k \left|\tilde{C}_{n-1}^3\right\rangle \end{aligned} \tag{14}$$

Since $\left|C_{n-1}^3\right\rangle$ and $\left|\tilde{C}_{n-1}^3\right\rangle$ are mutually orthogonal state, after Z-basis measurement on each qubit except qubit k, the sum of $\theta_1, \theta_2, \ldots, \theta_n$ can be determinately collapsed at aggregation node in the form of trigonometric functions.

Here we give an example in Fig. 9, to illustrate that one qubit 5 in C_5^3 state can obtain the sum of values $\theta_1, \theta_2, \theta_3, \theta_4$ assigned at other qubits.

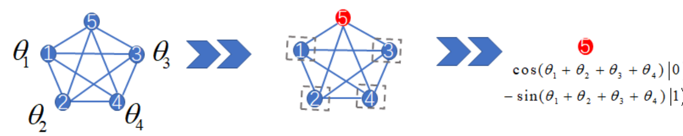

Fig. 9: Calculate the sum of $\theta_1, \theta_2, \theta_3, \theta_4$ assigned at four qubits in C_5^3 state by a aggregation node where qubit 5 located.

IV. Discussion and Conclusions

Multipartite entanglement is a fundamental resource in the Quantum Internet. In this paper, we identify four key functionalities for managing multipartite states non-locally with the communication overhead in view. A specific multipartite entangled state is proposed, which exhibits the attractive feature of consuming at most one EPR for the aforementioned management. And it also owns an active computing property for quantum distributing network.

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References

[1] M. Caleffi, A. S. Cacciapuoti, and G. Bianchi, "Quantum internet: From communication to distributed computing!" in *Proc. of ACM NANOCOM '18*. ACM, 2018, pp. 1–4.

[2] A. S. Cacciapuoti, M. Caleffi, F. Tafuri *et al.*, "Quantum internet: Networking challenges in distributed quantum computing," *IEEE Network*, vol. 34, no. 1, pp. 137–143, 2020.

[3] S. Pirandola and S. L. Braunstein, "Physics: Unite to build a quantum internet," *Nature*, vol. 532, no. 7598, pp. 169–171, 2016.

[4] S. Wehner, D. Elkouss, and R. Hanson, "Quantum Internet: a Vision for the Road Ahead," *Science*, vol. 362, no. 6412, 2018.

[5] W. Kozlowski, S. Wehner, R. V. Meter *et al.*, "Architectural principles for a quantum internet," Internet Engineering Task Force, Internet-Draft draft-irtf-qirg-principles-10, 2022, work in Progress.

[6] J. Illiano, M. Caleffi, A. Manzalini, and A. S. Cacciapuoti, "Quantum internet protocol stack: A comprehensive survey," *Computer Networks*, p. 109092, 2022.

[7] S. Koudia, A. S. Cacciapuoti, and M. Caleffi, "How deep the theory of quantum communications goes: Superadditivity, superactivation and causal activation," *IEEE Communications Surveys & Tutorials*, July.

[8] J.-C. Besse, K. Reuer, M. C. Collodo, A. Wulff, L. Wernli, A. Copetudo, D. Malz, P. Magnard, A. Akin, M. Gabureac, and et al., "Realizing a deterministic source of multipartite-entangled photonic qubits," *Nature Communications*, vol. 11, no. 1, Sep 2020.

[9] A. S. Cacciapuoti, S. Illiano, Jessica Koudia, and M. Caleffi, "The quantum internet: Enhancing classical services one qubit at a time," *IEEE Networks*, 2022.

[10] M. A. Nielsen, "Optical quantum computation using cluster states," *Phys. Rev. Lett.*, vol. 93, p. 040503, Jul 2004.