From the Environment-Assisted Paradigm to the Quantum Switch

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***Abstract*—The quantum switch has been witnessing growing attention in the last years due to its advantage in several quantum technologies applications. In particular, it has been proven that the quantum switch can significantly improve the communication rates beyond the limits of conventional quantum Shannon theory. In this paper, we theoretically prove that the quantum switch can be interpreted as a particular instance of the Environment-assisted quantum communication paradigm. The developed analysis is crucial to better understand the limitations of the quantum switch. Furthermore, the analysis is key to shed the light on control strategies within the Environment-assisted communication paradigm.**

***Index Terms*—Quantum communication, Quantum switch, Environment-assisted communication.**

I. Introduction

An important non-trivial approach for quantum communication protocols is based on quantum feedback control [1]. This approach relies on monitoring the environment, by measuring it after interaction with the considered quantum system, and accordingly it performs some corrections on the state of the information carrier, to retrieve it or completely restore it, by enhancing the communication rates. This is known as Environment-Assisted (EnA) quantum communication paradigm. Extensive work [2]–[7] has been done in this direction, both for discrete and continuous variable quantum systems, determining the optimal capacity of a given channel linking a sender (Alice) and a receiver (Bob).

Another important approach that has been witnessing growing attention in the last few years, is the one exploting the quantum switch [8]–[16]. Such a quantum switch is described mathematically by a supermap, which takes two (or more) channels and places them in a superposition of causal orders [17]. The information carrier in this setup propagates through the channels in an unusual quantum configuration, that differs from the known classical configurations where usually the channels are placed sequentially or in parallel. This is depicted in Fig. 1. This new configuration has been proved to provide advantages in many quantum applications such as quantum metrology [18], quantum computing [19] and quantum communication [15], [16], [20]. In particular with reference to the communication realm, it has been proved that the quantum switch can significantly improve the communication rates beyond the limits of conventional quantum Shannon theory. The advantage of the quantum switch is due to the genuinely quantum coherence between different operations, which does not exist in the classical realm.

In this paper, we analyze the quantum switch from a different perspective. Specifically, we theoretically prove that the quantum switch can be interpreted as a particular instance of the EnA quantum communication paradigm. To the best of our knowledge this is the first work addressing this issue.

The developed analysis is crucial to better understand the limitations of the quantum switch. Furthermore, the analysis is key to shed the light on control strategies within the EnA communication paradigm.

The paper is structured as follows. In Sec. II, we provide some preliminaries, needed for the developed analysis. In Sec. III we prove that the quantum switch can be seen as a particular instance of the Environment-assisted quantum communication paradigm. Finally in Sec. IV, we conclude the paper.

II. Preliminaries

In this section, we provide the preliminaries needed for the developed analysis.

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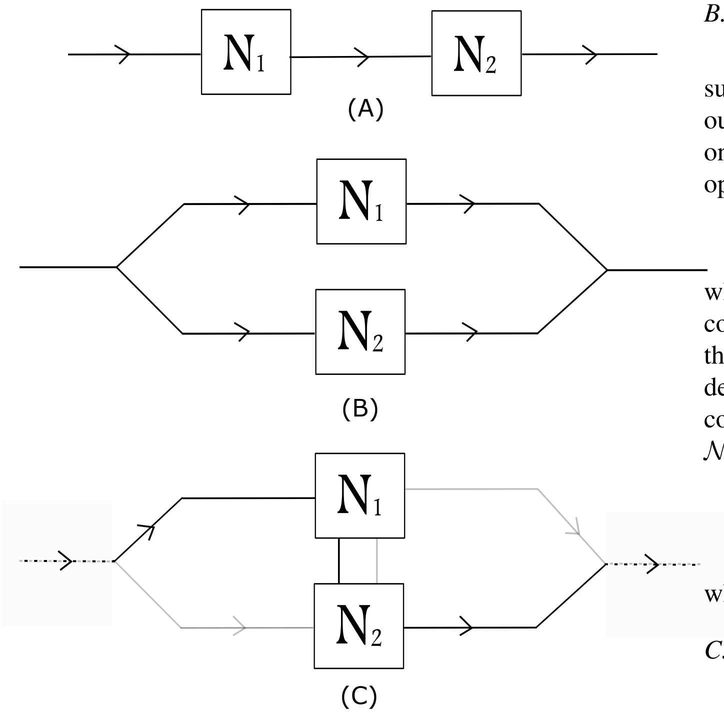

Fig. 1. (A) Classical configuration of channels $\mathcal{N}_1(\cdot)$ and $\mathcal{N}_2(\cdot)$ placed sequentially. (B) Classical configuration of the channels placed in parallel. (C) A quantum configuration realized by the quantum switch, where the information traverses the two channels in a coherent superposition of different orders.

A. Quantum channels

A quantum channel $\mathcal{N}(\cdot)$ can be described mathematically by a completely positive trace preserving map (CPTP) from the set of the input states belonging to the the input Hilbert space $\mathcal{H}^A$ to the set of the output states over the output Hilbert space $\mathcal{H}^B$. Specifically it is described by the Kraus decomposition as [21]:

$$\mathcal{N}(\rho) = \sum_i N_i \rho N_i^\dagger \tag{1}$$

where $\{N_i\}$ are known as Kraus operators, satisfying the equality $\sum_i N_i^\dagger N_i = \mathrm{I}$.

The quantum channel can also be described by its Stinespring dilation by some isometry V satisfying [22]

$$\mathcal{N}(\rho) = Tr_E(V \rho V) \tag{2}$$

where V is a map from $\mathcal{H}^A$ to $\mathcal{H}^B \otimes \mathcal{H}^E$, where E is called the environment. Such an isometry can be constructed from the Kraus decomposition of the channel as

$$V = \sum_i N_i \otimes |i\rangle_E \tag{3}$$

where $\{|i\rangle\}_E$ forms an orthogonal basis for the environment E.

B. The quantum switch

Mathematically, the quantum switch is described by a supermap taking two channels $\mathcal{N}_1(\cdot)$ and $\mathcal{N}_2(\cdot)$ as inputs, and outputs a channel in a superposition of orders of the original ones. Its action on quantum states is defined by the Kraus operators [8], [17]:

$$N_{ij}^{QS} = N_i^1 N_j^2 \otimes |0\rangle\langle 0| + N_j^2 N_i^1 \otimes |1\rangle\langle 1| \tag{4}$$

where $\{N_i^1\}$ and $\{N_i^2\}$, are the Kraus operators of the considered channels $\mathcal{N}_1(\cdot)$ and $\mathcal{N}_2(\cdot)$. The causal order of the communication channels is controlled by a quantum degree of freedom, represented by a control qubit ω. According to the quantum switch supermap, the output state $\mathcal{N}^{\mathcal{QS}}(\mathcal{N}_1, \mathcal{N}_2, \omega)(\rho)$ is given by:

$$\mathcal{N}^{\mathcal{QS}}(\mathcal{N}_1, \mathcal{N}_2, \omega)(\rho) = \sum_{ij} N_{ij}^{QS}(\rho \otimes \omega) \left(N_{ij}^{QS}\right)^\dagger, \tag{5}$$

where ρ is the input informational quantum state.

C. Environment-Assistance

In EnA communication protocols, it is assumed that the sender (Alice) and the receiver (Bob) have no access to the environment. Nevertheless, a third party (Charlie) who has access to it, can perform measurements $\{\Pi^x\}$, and communicate the obtained classical information x to Bob, who accordingly performs a correcting operation $\{R^x\}$ to recover the information sent by Alice. This constitutes a one way LOCC (Local Operation and Classical Communication) from Charlie to Bob and is described by the map:

$$\mathcal{L}^{C\to B} = \sum_x \Pi^x \otimes \mathcal{R}^x \tag{6}$$

where $\{\Pi^x\}_x$ is a set of CP maps, i.e., quantum instrument, performed by Charlie on the state of the environment, such that $\sum_x \Pi^x = \mathrm{I}$, and $\{\mathcal{R}^x\}_x$ is a set of CPTP maps performed by Bob according to the received measurement outcome "x". This approach is depicted in Fig. 2.

According to the EnA paradigm, the output state $N^{EnA}(\mathcal{N})(\rho)$ is given by

$$\mathcal{N}^{EnA}(\mathcal{N})(\rho) = (\mathcal{L}^{C\to B} \circ V)(\rho), \tag{7}$$

where V is the isometric extension of the channel $\mathcal{N}(\cdot)$.

III. From the Environment-Assistance Paradigm to the Quantum Switch

As mentioned in the previous section, in an EnA protocol it is assumed that the environment is controlled by a third party, Charlie, who is able to measure it in some bases and to communicate the outcomes to Bob. In turn, Bob tries to retrieve the information encoded in the quantum system by performing quantum maps conditioned on the received classical information from Charlie.

In order to show that the quantum switch can be seen as a special type of an environment-assisted communication

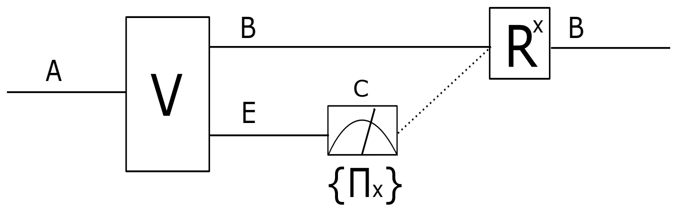

Fig. 2. A scheme depicting an environment-assisted communication protocol. Alice (A) wants to communicate quantum information to Bob (B) through a channel described by its isometric extension V. Charlie (C) who has access to the output environment of the communication channels performs a measurement Π_x on a given basis and communicated the classical outcome "x" to Bob, who performs a quantum operation $\mathcal{R}^x$ accordingly, to recover the state of the system.

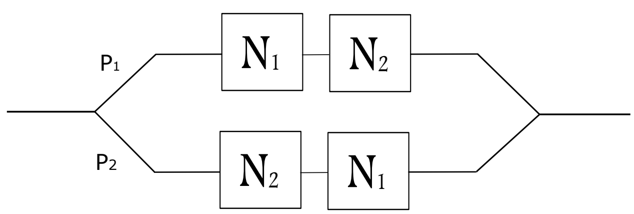

Fig. 3. A scheme depicting the channel $\Lambda(\cdot)$ in the text. The channel $\Lambda(\cdot)$ is a probabilistic mixture of different orders of the channels $\mathcal{N}_1(\cdot)$ and $\mathcal{N}_2(\cdot)$, with weighting probabilities p_1 and p_2

protocol, let us consider the convex combination of different orders of the channels $\mathcal{N}_1(\cdot)$ and $\mathcal{N}_2(\cdot)$, depicted in Fig. 3:

$$\Lambda(\rho) = p_1\mathcal{N}_1 \circ \mathcal{N}_2(\rho) + p_2\mathcal{N}_2 \circ \mathcal{N}_1(\rho), \tag{8}$$

where $\mathcal{N}_1 \circ \mathcal{N}_2(\cdot)$ and $\mathcal{N}_2 \circ \mathcal{N}_1(\cdot)$ denote the concatenation of the two channels in different orders, and $\sum_{i=1,2} p_i = 1$. This constitutes a classical configuration of channel placements. This channel along with precise measurement strategies on its environment will lead to a complete equivalence to the quantum switch map in (5).

By accounting for (8) it is possible to show the following result:

***Proposition** 1: There exists a measurement on a subspace E' of the Environment such that the EnA output state $\mathcal{N}^{EnA}(\mathcal{N}_1, \mathcal{N}_2)(\rho)$ is given by:*

$$\begin{aligned}&\mathcal{N}^{EnA}(\mathcal{N}_1, \mathcal{N}_2)(\rho) = \\ &\frac{1}{2}\sum_{ij}\{N_i^1, N_j^2\}\rho\{N_i^1, N_j^2\}^\dagger \otimes |+\rangle\langle+|_{E'} \\ &+ \frac{1}{2}\sum_{ij}[N_i^1, N_j^2]\rho[N_i^1, N_j^2]^\dagger \otimes |-\rangle\langle-|_{E'},\end{aligned} \tag{9}$$

where $\{N_i^1\}$ and $\{N_i^2\}$ are the Kraus operators of the considered channels $\mathcal{N}_1(\cdot)$ and $\mathcal{N}_2(\cdot)$.

Proof 1: See Appendix A

The following corollary could be derived from this proposition.

***Corollary** 1: The EnA output state $\mathcal{N}^{EnA}(\mathcal{N}_1, \mathcal{N}_2)(\rho)$ given in Proposition 1 coincides with the output state $\mathcal{N}^{QS}(\mathcal{N}_1, \mathcal{N}_2, \omega)(\rho)$ of the quantum switch supermap for the considered quantum channels $\mathcal{N}_1(\cdot)$ and $\mathcal{N}_2(\cdot)$,* when the control qubit is initiated in the state $\omega = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$.

Proof 2: see Appendix B

From Corollary 1, it results that in the case of the quantum switch, the role of the residual environment E' is played by the control qubit, leading to different effective channels, which result in the same form (9) if the control qubit is set to be in the maximally coherent state in the computational basis. Changing the initial state of the control qubit the switch is equivalent to different measurements on particular subspaces of the environment.

From the developed analysis, one can argue that the considered EnA-protocol substitutes the superposition of orders, characterizing the quantum switch supermap, with a classical combination of the orders (convex combination), combined with a full knowledge of the considered channels. On the contrary, the quantum switch supermap does not rely on a full knowledge of the considered channels but on the capabilities of the quantum particles to propagate according to quantum trajectories.

As a numerical example illustrating Proposition 1 and Corollary 1, let us consider the entanglement breaking channel described, analytically, by

$$\mathcal{N}_{XY}(\rho) = \frac{1}{2}(X\rho X + Y\rho Y), \tag{10}$$

with Kraus operators $\{\frac{1}{\sqrt{2}}X, \frac{1}{\sqrt{2}}Y\}$. It was proved in [13] that the output of the quantum switch in presence of entanglement breaking channels is given by:

$$\mathcal{N}^{QS}(\mathcal{N}_{XY}, \mathcal{N}_{XY}, \omega) = \frac{1}{2}\rho \otimes |+\rangle\langle+|_\omega + \frac{1}{2}Z\rho Z \otimes |-\rangle\langle-|_\omega \tag{11}$$

leading to perfect correction of the entanglement-breaking channel after the measurement outcomes on the coherent basis of the control qubit. The same correction can be performed by exploiting the considered EnA protocol in Proposition 1. To better see this, replacing the Kraus operators of the entanglement-breaking channel in (9) leaves us with the output state:

$$\mathcal{N}^{EnA}(\mathcal{N}_{XY}, \mathcal{N}_{XY}) = \frac{1}{2}\rho \otimes |+\rangle\langle+|_{E'} + \frac{1}{2}Z\rho Z \otimes |-\rangle\langle-|_{E'}. \tag{12}$$

If Charlie measures the outcome "+", it communicates such outcome to Bob who accordingly performs an identity channel on the system. If instead, Charlie measures outcome "−", it communicates this to Bob, who should perform a unitary given by the Z Pauli matrix. This results into a complete recovery of the information state ρ.

A density plot comparing the two maps in the case of the channel $\mathcal{N}_{XY}$ is depicted in Fig. 4 in terms of fidelity between the obtained output states. We consider an arbitrary input state ρ, given in the Bloch representation by

$$\rho = \frac{I}{2} + \frac{1}{2}(r_x X + r_y Y + r_z Z) \tag{13}$$

The fidelity is given by [23]

$$F(\rho, \sigma) = \mathrm{Tr}(\sqrt{\sqrt{\rho}\sigma\sqrt{\rho}})^2 \tag{14}$$

The fidelity reaches its maximal value for all the values of the Bloch coefficients of the state, validating the results of the theoretical analysis in Corollary 1.

The importance of this approach lies in the fact that it facilitates to understand why does the switch provide perfect correction of some channels which are unitarily equivalent to the entanglement breaking channel given in (10), whereas it fails to achieve such an advantage for other channels like the bit flip and the phase flip channels. It can also be used as a recipe for higher superposition of causal orders in the presence of many channels, to easily obtain the outcoming states of higher order quantum switches from simple fixed measurement strategies on the environment.

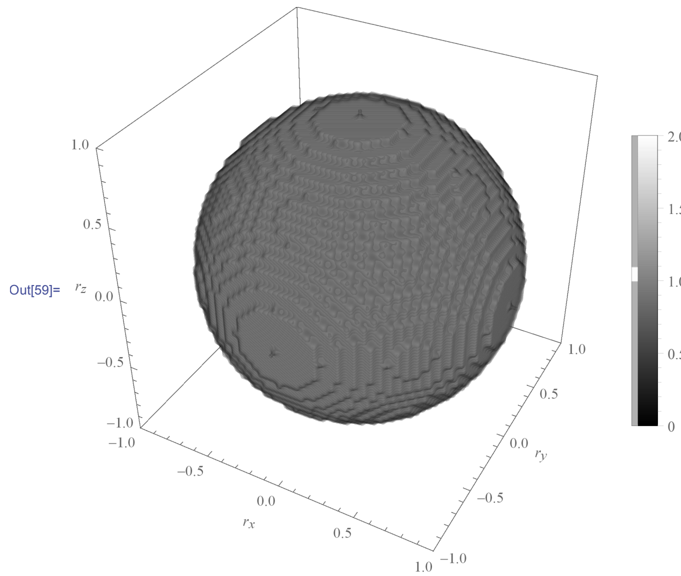

Fig. 4. A Density plot of the fidelity function between the output states resulting from the map S in Prop.(1) and the quantum switch map, on the Bloch sphere, for the channel $\mathcal{N}_{XY}$

IV. Conclusion

In this work, we have studied the quantum switch from the dynamical point of view of channels, as an isometric evolution of the quantum system together with its environment. We have shown that, the quantum switch can be thought as a particular instance of the environment-assisted communication paradigm. This can be useful, from the quantum communication point of view, to understand the reasons for which the quantum switch corrects perfectly some channels and fails to do so for others. It might also be of considerable importance for the study of the capacity achievable via the quantum switch beyond the single-shot capacity, and in different controlled orders beyond the two orders.

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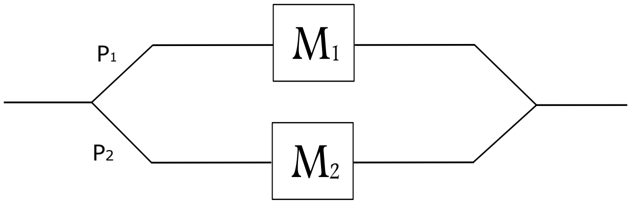

Fig. 5. A scheme depicting the channel $\Lambda'(\cdot)$. The channel $\Lambda'(\cdot)$ is a probabilistic mixture of the channels $\mathcal{M}_1(\cdot)$ and $\mathcal{M}_2(\cdot)$, with weighting probabilities p_1 and p_2

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Appendix A
Proof of Proposition 1

To prove Proposition.1, we let a quantum channel $\Lambda'(\cdot)$ be given by the convex combination of two channels $\mathcal{M}_1(\cdot)$ and $\mathcal{M}_2(\cdot)$ as is shown in Fig. 5, and given formally by

$$\Lambda'(\rho) = \sum_{i=1,2} p_i \mathcal{M}_i(\rho) \tag{15}$$

A possible Stinespring dilation of $\Lambda'(\cdot)$ can be given by the isometry V

$$V = \sum_k M_k^1 \otimes |\Gamma_k^1\rangle_E + \sum_{k'} M_{k'}^2 \otimes |\Gamma_{k'}^2\rangle_E \tag{16}$$

where $\{|\Gamma_{k,k'}^{1,2}\rangle_E\}$ are two independent degrees of freedom of the environment of the channel Λ, satisfying

$$\langle\Gamma_k^i|\Gamma_{k'}^{i'}\rangle_E = \delta_{ii'}\delta_{kk'}$$

It is easy to understand that each channel (noise) emerges dynamically from some coupling with the corresponding degrees of freedom of the environment that is given by the direct sum $E = \oplus_i \Gamma^i$. The action of the isometry V on a given state of a quantum system is

$$\begin{aligned} V\rho V^\dagger = &\sum_k M_k^1 \rho M_k^{1\dagger} \otimes |\Gamma_k^1\rangle\langle\Gamma_k^1|_E \\ &+ \sum_{k'} M_{k'}^2 \rho M_{k'}^{2\dagger} \otimes |\Gamma_{k'}^2\rangle\langle\Gamma_{k'}^2|_E \\ &+ \sum_{kk'} M_k^1 \rho M_{k'}^{2\dagger} \otimes |\Gamma_k^1\rangle\langle\Gamma_{k'}^2|_E \\ &+ \sum_{kk'} M_{k'}^2 \rho M_k^{1\dagger} \otimes |\Gamma_{k'}^2\rangle\langle\Gamma_k^1|_E \end{aligned} \tag{17}$$

We can see that tracing out the degrees of freedom of the environment leaves us with the channel $\Lambda'(\cdot)$.

The isometry describing the channel is unique up to an isometry on the environment, that is, for an appropriate isometry V given by (16), and a given isometry U on the environment defined by

$$\begin{aligned} U|\Gamma_k^1\rangle_E &= |\Gamma_k^1\rangle_E \otimes |0\rangle_{E'} \\ U|\Gamma_{k'}^2\rangle_E &= |\Gamma_{k'}^2\rangle_E \otimes |1\rangle_{E'} \end{aligned} \tag{18}$$

the map

$$V' = \sum_k M_k^1 \otimes |\Gamma_k^1\rangle_E \otimes |0\rangle_{E'} + \sum_{k'} M_{k'}^2 \otimes |\Gamma_{k'}^2\rangle_E \otimes |1\rangle_{E'} \tag{19}$$

is also an isometry of the channel Λ'. The isometry U on the environment is well defined and always exists. Let's assume that there is an agent who can control coherently the output environment of the channel Λ, in such a way he can measure E in the coherent basis given by $\sum_{kk'} c_k|\Gamma_k^1\rangle_E + c_{k'}|\Gamma_{k'}^2\rangle_E$, where $c_{k,k'} = \pm c$, with c an appropriate overall normalisation factor. The post-selected evolutions on each measurement outcome, are equivalent up to an isometry on E (up to minus signs), therefore the effective post-selected evolution of the system and the residual environment E' should be

$$\mathcal{N}^{EnA}(\mathcal{M}_1, \mathcal{M}_2) = \sum_k M_k^1 \otimes |0\rangle_{E'} + \sum_{k'} M_{k'}^2 \otimes |1\rangle_{E'} \tag{20}$$

The action of this postselected map is given by

$$\begin{aligned} &\mathcal{N}^{EnA}(\mathcal{M}_1, \mathcal{M}_2)(\rho) = \\ &\frac{1}{2}\sum_{kk'} (M_k^1 + M_{k'}^2)\rho(M_k^1 + M_{k'}^2)^\dagger \otimes |+\rangle\langle+|_{E'} \\ &+ \frac{1}{2}\sum_{kk'} (M_k^1 - M_{k'}^2)\rho(M_k^1 - M_{k'}^2)^\dagger \otimes |-\rangle\langle-|_{E'} \end{aligned} \tag{21}$$

If a helper, Charlie for instance, controlls the residual environment, he can perform a coherent measurement in the coherent basis $\{\frac{1}{\sqrt{2}}(|0\rangle \pm |1\rangle)\}$, upon which, the post-selected evolutions on the system are given by

$$\begin{aligned} \rho &\to \sum_{kk'} (M_k^1 + M_{k'}^2)\rho(M_k^1 + M_{k'}^2)^\dagger \\ \rho &\to \sum_{kk'} (M_k^1 - M_{k'}^2)\rho(M_k^1 - M_{k'}^2)^\dagger \end{aligned} \tag{22}$$

Let the channels $\mathcal{M}_1(\cdot)$ and $\mathcal{M}_2(\cdot)$ refer to the concatenation of two subsequent channels $\mathcal{N}_1(\cdot)$ and $\mathcal{N}_2(\cdot)$ in different orders, as is shown in Fig. 3 that is

$$\begin{aligned}\mathcal{M}_1(\cdot) &= \mathcal{N}_2 \circ \mathcal{N}_1(\cdot) \\ \mathcal{M}_2(\cdot) &= \mathcal{N}_1 \circ \mathcal{N}_2(\cdot)\end{aligned} \tag{23}$$

Replacing their explicit Kraus operators in Eq.(21), we get exactly the same form of Proposition.1

$$\begin{aligned}&\mathcal{N}^{EnA}(\mathcal{N}_1,\mathcal{N}_2)(\rho) = \\ &\frac{1}{2}\sum_{ij}\{N_i^1, N_j^2\}\rho\{N_i^1, N_j^2\}^\dagger \otimes |+\rangle\langle+|_{E'} \\ &+ \frac{1}{2}\sum_{ij}[N_i^1, N_j^2]\rho[N_i^1, N_j^2]^\dagger \otimes |-\rangle\langle-|_{E'}\end{aligned} \tag{24}$$

Appendix B
Proof of Corollary 1

By accounting for (4), and by assuming $\omega = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$, with some algebraic manipulations, it is possible to re-write eq. (5) as:

$$\begin{aligned}&\mathcal{N}^{\mathcal{QS}}(\mathcal{N}_1,\mathcal{N}_2,\omega) = \\ &\frac{1}{2}\sum_{ij}\{N_i^1, N_j^2\}\rho\{N_i^1, N_j^2\}^\dagger \otimes |+\rangle\langle+|_{\omega} \\ &+ \frac{1}{2}\sum_{ij}[N_i^1, N_j^2]\rho[N_i^1, N_j^2]^\dagger \otimes |-\rangle\langle-|_{\omega}\end{aligned} \tag{25}$$

with the subscript ω refering to the state of the control qubit. This is equivalent to Eq.(9), by exchanging the state of the residual environment in the case of the EnA protocol, with the control qubit in the quantum switch.